Minimum Weight Design of Finite Element Structures

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Design algorithms for obtaining the minimum weight of finite element structure are presented. The procedures are based upon approximating the structural behavior as a function of the reciprocal of the characteristic cross-sectional property of the finite elements. In addition, approximations based upon the compatibility equations are employed. This results in simpler and generally more accurate approximations of the deflection and stress changes as compared to present procedures which utilize the size variables and the equilibrium equations. Algorithms based upon linear programming utilizing linear constraints obtained either from the global equilibrium or compatibility equations plus a gradient search procedure with segmented linear constraints are developed. The efficiency of the linear programming with linear (equilibrium) constraints, which utilizes available digital sub-programs, was demonstrated by comparison of solutions of 3- and 10-bar truss problems presented in the literature. All solutions were of equal or less weight than published results and generally were obtained with fewer redesign cycles. The other algorithms are potentially more efficient. They require, however, further development and demonstration.

Nomenclature‡

U_N, λ = scalars
V_i = vector (i elements)
M_{ii} = square matrix
M_{if} = rectangular matrix (i rows, f columns)
D_i^i = diagonal matrix
$V^i V_i$ = dot product (scalar)
M_{ii}^{-1} = inverse of a square nonsingular matrix
$G_{if}^{-1} = (G_{fi} G_{if})^{-1} G_{fi}$ = pseudo (least square) inverse of a rectangular matrix ($i \leqq f$)
${}^{(n)}\chi_i, {}^{(n)}\lambda$ = nth solution (vector or scalar)
I_i^i = unit diagonal matrix
I_i = unit vector (all elements equal to unity)

I. Introduction

PRESENT procedures for designing finite element structures for minimum weight generally utilize an iteration technique which employs the global equilibrium equations to analyze and modify the structure. Although the global equilibrium equations, which are an equivalent condensation of the local equilibrium and compatibility equations, are most efficient for the analysis of a structure, they are somewhat wanting for design purposes. Optimum design procedures require a knowledge of how the structural behavior (e.g. deflections and stresses) changes as we change the elements. This is exceedingly difficult with solutions of the equilibrium equations since the size parameter is expressible as a diagonal matrix embedded in a product between rectangular direction cosine matrices G_{if} which must be inverted. The effect of element change upon structural behavior is approximated from the equilibrium equations with some gross approximations and mathematical manipulations as indicated in this paper. A more direct approach to approximating the structural behavior change is to utilize the compatibility equations. The "least square" solution of the compatibility equation leads to a simple and direct relationship between changes in structural response and the reciprocal of the cross-sectional size. This relationship is recommended for establishing approximations of structural behavior change for utilization in design procedures.

II. Technical Discussion

All design procedures utilize an initial guess as to the size A (e.g. area of rods, section modulus of beams, and thickness of plate) of the finite elements i of a structure of known geometry (e.g. global coordinates and resulting dimensions L). The structure is then analyzed to obtain the unrestrained deflections Δ_f and stresses σ_i for the design loadings P_f. Engineering judgement and/or a design algorithm is then utilized to resize the structure in order to obtain a lightweight design which does not violate limits upon the stresses (σ_{Ai}, allowable stresses), deflections (Δ_{Af}, allowable displacements), and size ($A_{\min} \leqq A \leqq A_{\max}$). This procedure is repeated until the last redesign is judged satisfactory.

Present techniques generally select a redesign based either upon an optimality criterion or a mathematical programming technique. The former approach requires the selection of criteria (e.g. fully stressed design, minimum energy, etc.) which it is hoped rapidly approaches the minimum weight design (MWD). Unfortunately the minimum weight design only coincides with a given optimality criterion for special conditions which are not necessarily met for a given problem. The latter approach selects lower weight redesigns but requires many redesigns and reanalyses because of the crudeness of the approximation of the resulting change in structural behavior. Since efficiency is inversely proportional to the number of reanalyses, the authors feel that a combination

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‡ Dummy subscripts and superscripts refer to the degrees of freedom associated with the element modes i, the element boundary modes m, and the global modes f. Note that the elements of a diagonal matrix and a vector are interchangeable, i.e.,

$$D_i^i X^i = X_i^i D_i \qquad D_i = I_i^i D_i = D_i^i I_i$$

and

$$\left(\frac{1}{X}\right)_i^i D_i^i X_i = \left(\frac{1}{X}\right)_i^i X_i^i D_i = I_i^i D_i = D_i$$

of both approaches, in which the optimality criteria is utilized to approach the MWD and a mathematical programming technique is employed to converge upon the MWD, would be more efficient than either approach. The authors also feel that the three algorithms suggested in this paper would demonstrate equal or greater efficiency. Each utilizes simple and relatively accurate approximations of structural behavior change and should result in rapid and accurate minimum weight designs. The first has been programmed combining available "stiffness" and "simplex" subroutines and the resulting solutions were compared to published solutions. Equal or greater structural and computer time efficiencies were indicated in all comparisons. The second design algorithm is also a linear programming procedure which should be of greater computer time efficiency than the first. The third design algorithm is a gradient search procedure which should be of greater accuracy and time efficiency than the first two, especially for structures with a large number of elements and loading conditions. It is strongly recommended that a digital program and demonstration examples of the third design algorithm be pursued.

Table 1 defines characteristic independent degrees of freedom i for several typical finite elements in terms of the "classical" boundary degrees of freedom m. The characteristic degrees of freedom are generally chosen so as to define allowable stresses. The degrees of freedom are related by the elemental equations of equilibrium

$$P_m = G_{mi} P_i \tag{1a}$$

and compatibility

$$\Delta_i = G_{im} \Delta_m \tag{1b}$$

These equations can be combined as indicated below

$$P_m = G_{mi} P_i = G_{mi} A_i^i \sigma_i = G_{mi} (AE)_i^i \varepsilon_i \tag{1c}$$

$$P_m = G_{mi} (AE)_i^i L_{ii}^{-1} \Delta_i = G_{mi} k_{ii} \Delta_i \tag{1d}$$

$$P_m = G_{mi} k_{ii} G_{im} \Delta_m = K_{mm} \Delta_m \tag{1e}$$

where G_{mi}, G_{im} are the equilibrium and compatibility matrices consisting of the direction cosines between the axes associated with the boundary and internal (characteristic) modes of load P and displacement Δ as indicated in Table 1. Associated with the elements are stress σ, strain ε, size A, L, and modulus E.

$$k_{ii} = (AE)_i^i L_{ii}^{-1} \tag{1f}$$

is a symmetrical characteristic stiffness matrix of the elements. $(AE)_i^i$ is a diagonal matrix of the "strain" stiffness of the finite elements whose variation (e.g. size, material, thermal, plasticity, etc.) define changes in structural behavior. L_{ii}^{-1} is a symmetrical matrix which is a function of the geometry of the elements (see Table 1), and is assumed invariant for small deflections.

Similar equations apply between the characteristic and global f degrees of freedom.

$$P_f = G_{fm} P_m = G_{fm} G_{mi} P_i = G_{fi} P_i \tag{2a}$$

$$\Delta_i = G_{im} \Delta_m = G_{im} G_{mf} \Delta_f = G_{if} \Delta_f \tag{2b}$$

$$P_f = G_{fi} k_{ii} \Delta_i = G_{fi} k_{ii} G_{if} \Delta_f = K_{ff} \Delta_f \tag{2c}$$

Algorithm Utilizing Equilibrium

The solution of the global equilibrium equation, Eq. (2c), is

$$\Delta_f = K_{ff}^{-1} P_f = [G_{fi} k_{ii} G_{if}]^{-1} P_f \tag{3a}$$

If the structure is modified, then from the condition

$$(K_{ff} + \delta K_{ff})(\Delta_f + \delta\Delta_f) = P_f \tag{3b}$$

it follows that

$$\delta\Delta_f = -(I_{ff} + K_{ff}^{-1}\delta K_{ff})^{-1} K_{ff}^{-1} \delta K_{ff} K_{ff}^{-1} P_f = -\left(\sum_{j=0}^{\infty} (-1)^j (K_{ff}^{-1}\delta K_{ff})^j\right) K_{ff}^{-1} \delta K_{ff} K_{ff}^{-1} P_f \tag{3c}$$

where

$$\delta K_{ff} = G_{fi} (\delta AE)_i^i L_{ii}^{-1} G_{if} = G_{fi} R_i^i k_{ii} G_{if} \tag{3d}$$

and

$$R_i = (\delta AE/AE)_i \tag{3e}$$

is the ratio of change of AE to original AE. A linear approximation ($j = 0$) of Eq. (3c) is

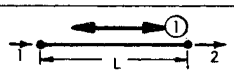
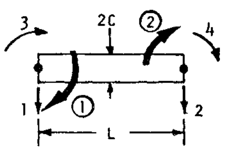
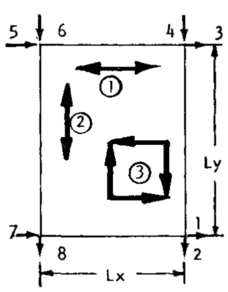
$$\begin{aligned}\delta\Delta_f &\sim K_{ff}^{-1} G_{fi} R_i^i k_{ii} G_{if} K_{ff}^{-1} P_f = K_{ff}^{-1} G_{fi} R_i^i P_i \\ &= K_{ff}^{-1} G_{fi} P_i^i R_i = K_{ff}^{-1} G_{fi} \left(\frac{P}{AE}\right)_i^i (\delta AE)_i \\ &= K_{ff}^{-1} G_{fi} (PB)_i^i (\delta AE)_i\end{aligned} \tag{3f}$$

where

$$B = 1/AE \tag{3g}$$

is the "strain" flexibility and

Table 1 Typical finite elements

ELEMENT	Characteristic Stresses and Strains (ϵ_i)	Diagram	Number of Degrees of Freedom i	m	G_{mi} ($P_m = G_{mi} P_i$)	L_{ii}^{-1} ($\epsilon_i = L_{ii}^{-1} \Delta_i$)	$(AE)_i^i = (B_i^i)^{-1}$ ($P_i = (AE)_i^i \epsilon_i$)
BAR	Axial (P/AE)		1	2	$\begin{Bmatrix}-1\\1\end{Bmatrix}$	$\frac{1}{L}$	AE
BEAM	Rotational $\frac{MC}{EI}$		2	4	$\begin{bmatrix}\frac{1}{L} & \frac{1}{L}\\ \frac{-1}{L} & \frac{-1}{L}\\ 1 & 0\\ 0 & 1\end{bmatrix}$	$\frac{2C}{L}\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}$	$\frac{EI}{C}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$
RECT. MEMBRANE PLATE	Membrane Strains (ϵ_x, ϵ_y, ϵ_{xy}) and stresses (σ_x, σ_y, σ_{xy})		3	8	$\alpha = L_y/L_x$ $\begin{bmatrix}1/\alpha & 0 & \alpha\\ 0 & \alpha & 1/\alpha\\ 1/\alpha & 0 & -\alpha\\ 0 & -\alpha & 1/\alpha\\ -1/\alpha & 0 & -\alpha\\ 0 & -\alpha & -1/\alpha\\ -1/\alpha & 0 & \alpha\\ 0 & \alpha & -1/\alpha\end{bmatrix}$	$\frac{1}{2(1-\nu^2) L_x L_y}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}$	$Et\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$

$$PB = \varepsilon \tag{3h}$$

is the characteristic strain.

Equation (3f) would suggest a linear interrelationship between the change of deflections and a change in the strain stiffness of a member. Further investigation (demonstrations and other algorithms) led to the conclusion that consideration of additional nonlinear terms should result in a similar expression utilizing the change in elemental flexibility δB. This should have been obvious since the deformations are linearly related to the flexibility matrix B and only inversely related to the stiffness matrix AE.

Consider the Taylor expansion of the change in the strain, and its linear approximations

$$\delta\varepsilon = \delta(P/AE) = \{-(P/AE)+[\partial P/\partial(AE)]\}[\delta(AE)/AE]+ \{(P/AE)-[\partial P/\partial(AE)]+(AE/2)[\partial^2 P/\partial(AE)^2]\} \times \{[\delta(AE)/AE]\}^2+\cdots \sim -\varepsilon\,\delta(AE)/AE \tag{4a}$$

and

$$\delta\varepsilon = \delta(PB) = (P+B\,\partial P/\partial B)\delta B + [(\partial P/\partial B)+(B/2)(\partial^2 P/\partial B^2)]\,\delta B^2+\cdots \sim \varepsilon\,\delta B/B \tag{4b}$$

The linear approximation of Eq. (4b) is exact when δP is identically zero while the linear approximation of Eq. (4a) remains an approximation which can be in gross error when P is large. Substitution of Eq. (3g) and the linear approximation of Eqs. (4) into Eq. (3f), results in

$$\delta\Delta_f \sim K_{ff}{}^{-1}G_{fi}(AE)_i^i\,\delta\varepsilon_i \sim K_{ff}{}^{-1}G_{fi}P_i^i\left(\frac{\delta B}{B}\right)_i \tag{5a}$$

and

$$\Delta_f+\delta\Delta_f \sim K_{ff}{}^{-1}G_{fi}P_i^i[1+(\delta B/B)]_i = K_{ff}{}^{-1}G_{fi}P_i^iX_i \tag{5b}$$

where

$$X = 1+(\delta B/B) = (AE/AE+\delta AE) \geq 0 \tag{5c}$$

For a linear programming design algorithm, the limiting deflections, stresses and sizes must be defined by a set of constraint equations which are linear with the variable X which is never negative. The constraints are therefore defined as follows: deflection

$$\Delta_f+\delta\Delta_f = K_{ff}{}^{-1}G_{fi}P_i^iX_i \leqq \Delta_{Af} \tag{6a}$$

stress

$$\sigma_i+\delta\sigma_i = E_i^iL_{ii}{}^{-1}G_{if}K_{ff}{}^{-1}G_{fi}P_i^iX_i \leqq \sigma_{Ai} \tag{6b}$$

size

$$X_i \leqq \left(\frac{AE}{AE_{\min}}\right)_i = (X_{\max})_i \tag{6c}$$

Appropriate sign change must be made in the constraint equations for the sense of the deflection and stresses by the sign function. Linear stress constraints based upon stability or biaxiality can be readily approximated, as indicated in Ref. 1. Note that all types of constraints are readily defined with equal ease and form, with the aid of the X variable and previously calculated matrices.

The utilization of design variables other than size has been suggested by several investigators. As examples, Ref. 2 discusses the reciprocal of size variable B, Ref. 3 employs the ratio of new to old size $1/X$, and Ref. 4 suggests the stress variable σ.

The weight (objective function) is not linear with the X design variable, but linear approximations of the objective function are readily obtained.

Let

$$W = \Sigma w_i\,X_i{}^{-e_i} = w^i(1/X^e)_i = (L\rho A^e)^i(1/X^e)_i \tag{7a}$$

then

$$\nabla W = \frac{\partial W}{\partial X_i} = -(ew/X^{e+1})_i = -(ew)_i \text{ at initial design } (X=1) \tag{7b}$$

where w is the initial weight of the member and e is an appropriate exponent which approximates the relationship between the weight and the size parameter ($e = 1$ for a uniform bar or plate and a constant depth tubular beam, $e \sim 3/2$ for uniform structural beams since A is section modulus).

The first design algorithm establishes the design of least weight, which exists at the intersection (vertex) of i constraints, by moving in the negative gradient direction to successive vertices of diminishing weight (see Fig. 1). If a linear programming algorithm results in an MWD design, then it is also a fully constrained design (FCD) where the i undefined sizes satisfy at least i constraints by equalities.

Algorithm Utilizing Compatibility

One of the deficiencies of the first design algorithm defined by Eqs. (6) and (7) is that the global flexibility $K_{ff}{}^{-1}$ is required to establish the constraints for each redesign. This can be time consuming, especially if we have a large number of degrees of freedom f. The redesign procedure can be simplified if we employ the least square solution of compatibility, Eq. (2b), to approximate the constraints. The most probable values of the global deflections Δ_f set for a given set of internal deflections Δ_i, is

$$\Delta_f \stackrel{2}{=} (G_{fi}G_{if})^{-1}G_{fi}\Delta_i = D_{fi}L_{ii}(PB)_i = F_{fi}\varepsilon_i \tag{8a}$$

Equation (8a) is exact when the assumed Δ_i and ε_i are compatible sets. This is equivalent to satisfying

$$I_i^i\Delta_i = \Delta_i \stackrel{2}{=} G_{if}D_{fi}\Delta_i = \Phi_{ii}\Delta_i \tag{8b}$$

or

$$\begin{aligned}\sigma_i &= E_i^iL_{ii}{}^{-1}\Phi_{ii}L_{ii}\varepsilon_i = T_{ii}\varepsilon_i \\ &= T_{ii}E_i^i\sigma_i = S_{ii}\sigma_i\end{aligned} \tag{8c}$$

Thus Eqs. (8) are satisfied by the original analysis and will hold for a correct determination of the changes in the global displacements and internal deflections or stresses, i.e.,

$$\sigma_i+\delta\sigma_i = T_{ii}(\varepsilon+\delta\varepsilon)_i = T_{ii}(P+\delta P)_i^i(BX)_i = T_{ii}(PB)_i^iX_i + T_{ii}(BX)_i^i\,\delta P_i \tag{9a}$$

and

$$\Delta_f+\delta\Delta_f = D_{fi}L_{ii}(E_i^i)^{-1}(\sigma+\delta\sigma)_i = F_{fi}(\varepsilon+\delta\varepsilon)_i \tag{9b}$$

The design philosophy is to estimate the stress and deflection changes resulting from a weight reducing design change, assuming Eqs. (9) are applicable. A more accurate approximation of the structural behavior after a size or modulus change is presented in the Appendix.

In the linear programming algorithm the first terms of Eqs. (9) are employed for the linear approximation of the constraint equations. The approximation would be exact if $\delta P \equiv 0$, which would correspond to a static determinate structure or a proportional resizing of each member (X is a constant), i.e.,

$$\sigma_i+\delta\sigma_i = T_{ii}(\sigma/E)_i^iX_i \leqq \sigma_{Ai} \tag{10a}$$

and

$$\Delta_f+\delta\Delta_f = F_{fi}(\sigma/E)_i^iX_i \leqq \Delta_{Af} \tag{10b}$$

These equations, however, are more accurate than Eqs. (6) for larger or proportional δP size changes. The design algorithm is identical to the previous one with the substitution of Constraint Eqs. (10a) and (10b) for Eqs. (6b) and (6a), respectively. Note that Eqs. (10) require the simple inversion of $G_{fi}G_{if}$ which is a sparse, well conditioned matrix with all nonzero terms concentrated near the main diagonal. Since the matrices G_{if}, T_{ii} and F_{fi} are invariants of the structure, they need be calculated only once and stored for each redesign.

The linear approximation of the stress and deflection changes was employed in the first two algorithms because the changes in internal loads δP_i are nonlinear and unknown, changing with each design change. The linear approximation algorithms result in a satisfactory minimum weight design because each redesign tends to approach the final MWD, and the linear approximation of the constraints is excellent in the vicinity of the final redesign ($X_i \sim 1$). The δP_i load system is self-equilibrating and acts as a perturbation upon the linear approximation converging more rapidly to zero than X converges to one. In addition, experience has demonstrated a very flat slope to the weight function in the

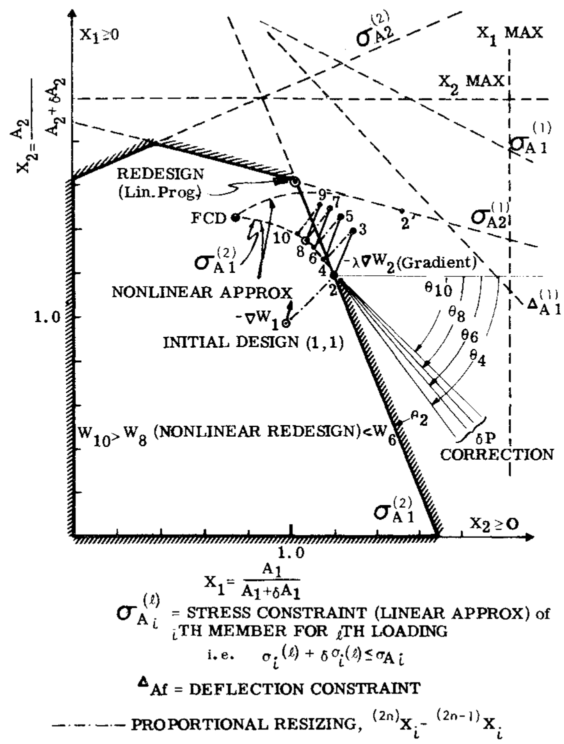


Fig. 1 Two-dimensioned representation of design algorithms.

vicinity of the MWD. Thus all designs in the immediate vicinity of the MWD are practically of equal weight. Another characteristic of the linear approximation algorithms is that the variability of the weight gradient, Eq. (7b), results in an oscillation of the weights of the redesigns about the MWD if it does not satisfy at least i constraints (FCD). The degree of oscillating can be made as small as desired by appropriate design change limits. The characteristics of linear approximation solutions of the MWD are presented in Ref. 1.

The first two design algorithms are inefficient due to the relatively large number of reanalyses and the fact that the computer time and storage requirements increases at a much greater rate than the number of elements and loading conditions. To overcome this deficiency, a general purpose design algorithm which should be more efficient because of reduced number of reanalyses and core requirements was developed from Eqs. (9). When we change a design by ${}^{(n)}\delta X_i = {}^{(n+1)}X_i - {}^{(n)}X_i$, the stresses and deflections change as indicated in Eqs. (9). The value of ${}^{(n+1)}\delta P_i$, in the third design algorithm, is approximated from the ${}^{(n)}\delta P_i$ that can be calculated from the stress change ${}^{(n)}\delta\sigma_i$ resulting from the nth value of the design vector ${}^{(n)}X_i$, i.e.,

$$ {}^{(n)}\delta P = (\sigma + {}^{(n)}\delta\sigma)(A + {}^{(n)}\delta A) - P \tag{11a} $$

Substituting Eq. (11a) into Eqs. (9) results in

$$ (\sigma + {}^{(n+1)}\delta\sigma)_i = T_{ii}(E_i^{\,i})^{-1}\left(\frac{\sigma + {}^{(n)}\delta\sigma}{{}^{(n)}X}\right)_i^{i} {}^{(n+1)}X_i \tag{11b} $$

and

$$ (\Delta + {}^{(n+1)}\delta\Delta)_f = F_{fi}(E_i^{\,i})^{-1}\left(\frac{\sigma + {}^{(n)}\delta\sigma}{{}^{(n)}X}\right)_i^{i} {}^{(n+1)}X_i \tag{11c} $$

Equations (11) are equivalent to an incremental improvement of the linear approximation as the design is incrementally changed in order to better approximate the nonlinear constraints. It can be visualized in Fig. 1, as a rotation ($\delta\theta$) of the constraint about the intersection (point 2) of the constraint with the radial extension of the initial design (point 1).

The design algorithm proceeds in the following manner:

1) An initial design (point 1, ${}^{(1)}X_i = 1_i$) is selected and analyzed for the stresses $\sigma_i^{\,l}$ and deflections $\Delta_f^{\,l}$ for each loading $P_f^{\,l}$.

2) The design is proportionally resized to point 2

$$ {}^{(2n)}X_i = {}^{(2n)}C\,{}^{(2n-1)}X_i \tag{12a} $$

where

$$ {}^{(2n)}C = \operatorname{Min}\left(\frac{\sigma_{Ai}}{\sigma_i^{\,l} + {}^{(2n-1)}\delta\sigma_i^{\,l}};\ \frac{\Delta_{Af}}{\Delta_f^{\,l} + {}^{(2n-1)}\Delta_f^{\,l}}\right) > 0 \tag{12b} $$

Note that a proportional change ${}^{(2n)}X/{}^{(2n-1)}X = {}^{(2n)}C$ results in proportional changes of the stresses and deflections with $\delta P \equiv 0$, i.e.,

$$ \Delta_f + {}^{(2n)}\delta\Delta_f = {}^{(2n)}C(\Delta_f + {}^{(2n-1)}\delta\Delta_f) \tag{12c} $$

$$ \sigma_i + {}^{(2n)}\delta\sigma_i = {}^{(2n)}C(\sigma_i + {}^{(2n-1)}\delta\sigma_i) \tag{12d} $$

The weight of the proportionally resized structure is calculated

$$ {}^{(2n)}W = \Sigma(w/{}^{(2n)}X^e)_i \tag{12e} $$

3) The design vector is incremented, staying within the size constraints, a small constant length λ in the negative gradient direction $[-(\partial W/\partial X_i) = (ew/X^{e+1})_i]$

$$ {}^{(2n+1)}X = {}^{(2n)}X + \lambda(ew/X^{e+1})_i \leqq \left(\frac{AE}{AE_{\min}}\right)_i = (X_{\max})_i \tag{12f} $$

4) The new stresses and deflections are calculated by the approximations of Eqs. (11b) and (11c).

5) Steps 2 and 3 are repeated until the resized weights start to oscillate.

$$ {}^{(2n-2)}W > {}^{(2n)}W \leqq {}^{(2n+2)}W \tag{12g} $$

6) Repeat steps 1–5 with a new initial design $(A/{}^{(2n)}X)_i$ and a much smaller value of λ. The new initial design should be in the immediate vicinity of a minimum weight design (MWD) and the approximations should be quite accurate because of the very small changes in δP_i. The weight changes should be quite small and a final design should result by the second conclusion of step 5, although it may be repeated as many times as required. Thus a final design can be obtained in as little as two reanalyses and only a few matrices need be stored and operated upon.

III. Examples

A number of 3- and 10-bar planar truss design problems (Fig. 2) were solved employing a digital program which incorporated the "stiffness" and "simplex" subprograms. The

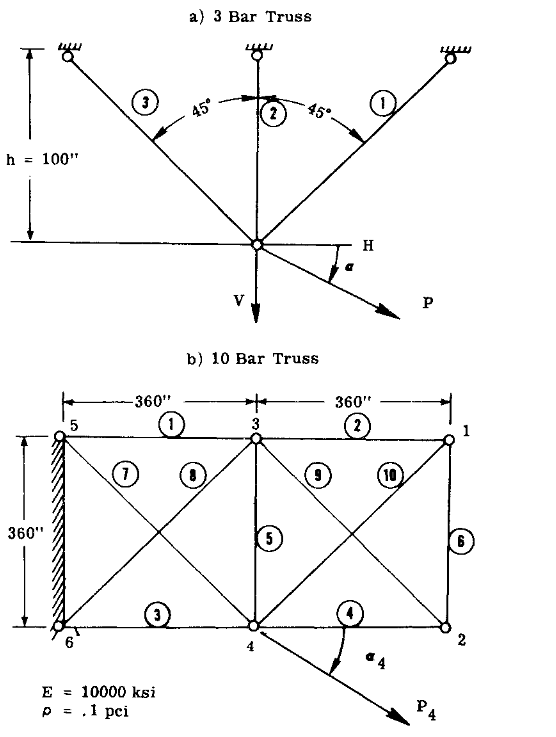


Fig. 2 Planar trusses: a) 3-bar truss, and b) 10-bar truss.

Table 2 3-Bar problems

Prob. No.	Loadings: P kips	Loadings: α degrees	Allowables: $\sigma_{A_1} = \sigma_{A_3}$ ksi	Allowables: σ_{A_2} ksi	Allowables: Δ_{AV} in.	Allowables: $A_{\min}$ sq. in.	Solutions: Ref. 1 Analytic	Solutions: Ref. 1 Digital	Solutions: No. of des. cycles	Solutions: Pub.	Solutions: Ref.
1	20	45	±20	±20	...	...	26.389	26.3897	9	26.39	5
	20	135								26.41	
2	20	45	±20	±20	±0.1	...	28.282	28.284	3	28.45	6
	20	135									
3	40	45	±5	±20	...	...	...	159.854	19	159.86	7
	30	90								159.91	8
	20	135								161.9	9
4a	20	45	±40	±40	±0.1	0.01	20.199	20.199	2	20.20	10
	20	135									
4b	20	22.5	±40	±40	±0.1	0.01	15.571	15.571	3	...	...
	20	155.5									
4c	20	67.5	±40	±40	±0.1	0.01	21.758	21.758	4	...	...
	20	112.5									
4d	20	90	±40	±40	±0.1	0.01	20.212	20.212	11	...	...
5	20	75	±20	±20	...	0.001	17.424	17.424	3	...	...
	20	105									
6	5	0	±5	±20	±0.1	0.001	...	25.00	3	...	...
	10	90									

program and solutions are presented in Ref. 1. The design search is based upon utilizing the linearized constraints and gradient [Eqs. (5) and (7)], calculated from an initial design of unit bar areas, to establish the lightest fully constrained design (FCD) which is then employed as a new initial design. This is continued until the redesigned weights converge or begin to oscillate. The oscillation about the minimum weight design (MWD) can be made as small as desired by reducing the design change limit for subsequent redesigns. The improvement in accuracy of the linear approximation of changes in structural behavior for very small size changes assures a rapid convergence to the immediate neighborhood of an MWD.

The solutions obtained are compared, in Tables 2 and 3, to published solutions and to closed form analytical solutions in Ref. 1, which were available. In all cases, the solutions were of equal or less weight than published values, and extremely close to analytical solutions. The number of redesigns required were generally smaller, indicating that the computation time on equivalent digital machines would be shorter.

Appendix

Approximate Static and Dynamic Behavior of Modified Structure

A Rayleigh-Ritz procedure which minimizes the total potential energy of two characteristic internal deflection patterns is employed to obtain an approximation of the change in static or dynamic structural response when the structure is modified.

$$[k_{ii} \to (1+R)_i^i k_{ii} = Y_i^i k_{ii}] \tag{A1a}$$

The first characteristic deflection pattern (ζ_i and ζ_f) is that of the unmodified structure which satisfies compatibility

$$\zeta_i = G_{if}\,\zeta_f \tag{A1b}$$

but may not satisfy equilibrium of the modified structure

$$P_f \stackrel{?}{=} G_{fi}\, Y_i^i k_{ii}\, \zeta_i \tag{A1c}$$

The second characteristic deflection pattern corresponds to zero change of original internal loads

$$Q_i = k_{ii}\,\zeta_i = k_{ii}\, Y_i^i X_i^i \zeta_i \tag{A1d}$$

in which each original internal displacement is divided by the ratio of the modified to unmodified internal stiffnesses of the members, i.e.,

$$\eta_i = \left(\frac{1}{Y}\right)_i^i \zeta_i = X_i^i \zeta_i \tag{A2a}$$

This deflection pattern results in equilibrium with the applied load

$$P_f = G_{fi}\, Y_i^i k_{ii}\, X_i^i \zeta_i \tag{A2b}$$

but may not satisfy compatibility with the associated global displacement

Table 3 10-Bar problems—single loading

Prob. No.	Loading: P kips	Loading: α degrees	Allowables: σ_A ksi	Allowables: Δ_A in.	Allowables: $A_{\min}$ sq. in.	Solutions: Digital[a]	Solutions: No. of des. cycles	Solutions: Published[b]
1	$P_2 = P_4 = 100$	$\alpha_2 = \alpha_4 = 90$	±25	...	0.1	1593.4[c]	5	1593.4
2	$P_2 = P_4 = 100$	$\alpha_2 = \alpha_4 = 90$	±25	±2.0	0.1	5080.4	15	5084.9
3	$P_2 = P_4 = 150$	$\alpha_2 = \alpha_4 = 90$	±25	...	0.1	1664.5[c]	5	1664.5
	$P_1 = P_3 = -50$	$\alpha_1 = \alpha_3 = 90$						
4	$P_2 = P_4 = 150$	$\alpha_2 = \alpha_4 = 90$	±25	±2.0	0.1	4758.4	12	4895.6
	$P_1 = P_3 = -50$	$\alpha_1 = \alpha_3 = 90$						
5	$P_2 = 100$	$\alpha_2 = 90$	±25	...	0.1	1161.2[c]	5	...
6	$P_4 = 100$	$\alpha_2 = 90$	±25	...	0.1	458.42[c]	5	...
7	$P_1 = 100$	$\alpha_1 = 0$	±25	...	0.1	603.56	5	...
	$P_2 = 100$	$\alpha_2 = 0$	±25	...	0.1			

[a] Digital solutions presented in Ref. 1.
[b] Published solution presented in Ref. 8.
[c] Solutions are obviously correct. For a single loading, all nonminimum area rods are stressed to allowable stress.[1]

$$\eta_f = T_{fi}\eta_i \quad \text{(A2c)}$$

$$\eta_i \stackrel{2}{=} G_{if}\,T_{fi}\eta_i = \Phi_{ii}\eta_i \quad \text{(A2d)}$$

Let the final displacements and internal loads be

$$\Delta_i = a\zeta_i + b\eta_i = (a + bX_i^{\,i})\zeta_i \quad \text{(A3a)}$$

$$\Delta_f = a\zeta_f + b\eta_f = a\zeta_f + bT_{fi}X_i^{\,i}\zeta_i \quad \text{(A3b)}$$

and

$$P_i = ak_{ii}Y_i^i\zeta_i + bk_{ii}Y_i^iX_i^{\,i}\zeta_i = (aY_i^i + b)Q_i \quad \text{(A3c)}$$

where a and b are undefined coefficients whose values are established by making the total potential energy stationary, i.e.,

$$\delta U = P^f\delta\Delta_f - P^i\delta\Delta_i = 0 \quad \text{(A4a)}$$

From Eqs. (A3)

$$\delta U = \delta a[U_0 - aU_2 - bU_0] + \delta b[U_1 - aU_0 - bU_3] \quad \text{(A4b)}$$

where

$$U_0 = P^f\zeta_f = Q^i\zeta_i \quad \text{(A5a)}$$

$$U_1 = P^f\eta_f = P^fT_{fi}X_i^{\,i}\zeta_i = P^fT_{fi}\zeta_i^{\,i}X_i = S^iX_i \quad \text{(A5b)}$$

$$U_2 = Q^iY_i^i\zeta_i = Q^i\zeta_i^{\,i}Y_i = V^iY_i \quad \text{(A5c)}$$

$$U_3 = Q^iX_i^{\,i}\zeta_i = Q^i\zeta_i^{\,i}X_i = V^iX_i \quad \text{(A5d)}$$

For stationary total potential energy

$$aU_2 + bU_0 = U_0 \quad \text{(A6a)}$$

$$aU_0 + bU_3 = U_1 \quad \text{(A6b)}$$

The solution of the coefficients is

$$a = \frac{U_3U_0 - U_0U_1}{U_1U_3 - U_0^{\,2}} = \frac{U_0[V^i - S^i]X_i}{(V^iY_i)(V^iX_i) - U_0^{\,2}} \quad \text{(A6c)}$$

$$b = \frac{U_2U_1 - U_0^{\,2}}{U_2U_3 - U_0^{\,2}} = \frac{(V^iY_i)(S^iX_i) - U_0^{\,2}}{(V^iY_i)\,V^iX_i) - U_0^{\,2}} \quad \text{(A6d)}$$

An unchanged structure ($R = 0, Y = 1, X = 1$) results in $a + b = 1$ and the original solution

$$\Delta_i = \zeta_i; \quad \Delta_f = \zeta_f; \quad P_i = Q_i$$

A proportional change in all the members (Y = constant) results in $b = 1 - Ya$ and an inverse proportional change in the deflections with no change in the internal load

$$\Delta_i = \zeta_i/Y; \quad \Delta_f = \zeta_f/Y; \quad P_i = Q_i$$

Thus the recommended approximation procedure results in exact solutions for those special cases for which generalized solutions exists.

For dynamic problems

$$\lambda M_f^{\,f}\zeta_f = K_{ff}\zeta_f \quad \text{(A7a)}$$

and

$$(\lambda + \delta\lambda)M_f^{\,f}\Delta_f = G_{fi}Y_i^ik_{ii}G_{if}\Delta_f \quad \text{(A7b)}$$

Letting

$$U_0 = \zeta^fM_f^{\,f}\zeta_f = \zeta^ik_{ii}\zeta_i \quad \text{(A8a)}$$

$$U_1 = \zeta^fM_f^{\,f}\eta_f = \zeta^fM_f^{\,f}T_{fi}\zeta_i^{\,i}X_i = D^iX_i \quad \text{(A8b)}$$

$$U_2 = \zeta^ik_{ii}Y_i^i\zeta_i = \zeta^ik_{ii}\zeta_i^{\,i}Y_i = H^iY_i \quad \text{(A8c)}$$

$$U_3 = \zeta^ik_{ii}X_i^{\,i}\zeta_i = \zeta^ik_{ii}\zeta_i^{\,i}X_i = H^iX_i \quad \text{(A8d)}$$

and

$$U_4 = \eta^fM_f^{\,f}\eta_f = \zeta^iX_i^{\,i}T_{if}M_f^{\,f}T_{fi}X_i^{\,i}\zeta_i$$
$$= X^i\zeta_i^{\,i}T_{if}M_f^{\,f}T_{fi}\zeta_i^{\,i}X_i = X^iN_{ii}X_i \quad \text{(A8e)}$$

then

$$\lambda + \delta\lambda = \frac{U_2 + ZU_0}{U_0 + ZU_1} = \frac{U_0 + ZU_3}{U_1 + ZU_4} \quad \text{(A9a)}$$

where

$$Z = \frac{a}{b} = \frac{U_6 \pm (U_6^{\,2} - 4U_5U_7)^{1/2}}{2U_5} \quad \text{(A9b)}$$

$$U_5 = U_0U_4 - U_1U_3 \quad \text{(A9c)}$$

$$U_6 = U_0U_3 - U_2U_4 \quad \text{(A9d)}$$

$$U_7 = U_0U_4 - U_1U_3 \quad \text{(A9e)}$$

defines the new eigenvalue $(\lambda + \delta\lambda)$ and the new eigenvector

$$\Delta_f = \zeta_f + T_{fi}\zeta_i^{\,i}X_i/Z \quad \text{(A9f)}$$

The "influence coefficients" for the static or dynamic changes caused by modifying the jth member [k_{jj} to $(1+\delta Y)k_{jj}$ or $k_{jj}^{\,-1}$ to $(1+\delta X)k_{jj}^{\,-1}$] can be evaluated in Eqs. (A6) and (A9) by noting that

$$\partial U_0/\partial Y = 0 \quad \text{(A10a)}$$

$$\partial U_1/\partial X = -P^fT_{fj}\Delta_j = -\lambda\zeta^fM_f^{\,f}T_{fj}\zeta_j \quad \text{(A10b)}$$

$$\partial U_2/\partial X = \partial U_3/\partial X = (P_j)(\zeta_j) \quad \text{(A10c)}$$

$$\partial U_4/(\partial X)^2 = \Delta^jT_{jf}M_f^{\,f}T_{fj}\zeta_j \quad \text{(A10d)}$$

Equations (A6) can be readily adapted to the gradient search procedure described in the paper. Equations (9b) and (9c) become

$$\left(\frac{\sigma + {}^{(n+1)}\delta\sigma}{\sigma}\right)_i = {}^{(n+1)}a + {}^{(n+1)}b\,{}^{(n+1)}X_i \quad \text{(A11a)}$$

and

$$(\Delta + {}^{(n+1)}\delta\Delta)_f = {}^{(n+1)}a\Delta_f + {}^{(n+1)}b(T_{fi}\Delta_i^{\,i})^{(n+1)}X_i \quad \text{(A11b)}$$

where ${}^{(n+1)}a$ and ${}^{(n+1)}b$ are defined by Eqs. (A6c) and (A6d) for ${}^{(n+1)}X_i$.

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